14-3 Trigonometric Identities

Main Ideas

- Use identities to find trigonometric values.
- Use trigonometric identities to simplify expressions.

New Vocabulary

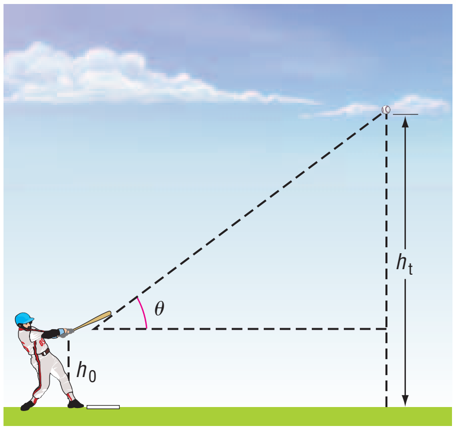
trigonometric identity

GET READY for the Lesson

A model for the height of a baseball after it is hit as a function of time can be determined using trigonometry. If the ball is hit with an initial velocity of v feet per second at an angle of θ from the horizontal, then the height h of the ball after t seconds can be represented by

$$h = \left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0,$$

where h_0 is the height of the ball in feet the moment it is hit.



Find Trigonometric Values In the equation above, the second term $\left(\frac{\sin\theta}{\cos\theta}\right)t$ can also be written as $(\tan\theta)t$. $\left(\frac{\sin\theta}{\cos\theta}\right)t = (\tan\theta)t$ is an example of a trigonometric identity. A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

The identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is true except for angle measures such as 90°, 270°, 450°, …, 90° + 180° · k. The cosine of each of these angle measures is 0, so none of the expressions tan 90°, tan 270°, tan 450°, and so on, are defined. An identity similar to this is $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

These identities are sometimes called *quotient identities*. These and other basic trigonometric identities are listed below.

KEY CONCEPT *Basic Trigonometric Identities*

Quotient Identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cos\theta \neq 0$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sin\theta \neq 0$	
Reciprocal Identities	$\csc\theta = \frac{1}{\sin\theta}$ $\sin\theta \neq 0$	$\sec\theta = \frac{1}{\cos\theta}$ $\cos\theta \neq 0$	$\cot\theta = \frac{1}{\tan\theta}$ $\tan\theta \neq 0$
Pythagorean Identities	$\cos^2\theta + \sin^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	
	$\cot^2\theta + 1 = \csc^2\theta$		

You can use trigonometric identities to find values of trigonometric functions.

EXAMPLE 1 Find a Value of a Trigonometric Function

a. Find $\cos\theta$ if $\sin\theta = -\frac{3}{5}$ and $90° < \theta < 180°$.

$\cos^2\theta + \sin^2\theta = 1$	Trigonometric identity
$\cos^2\theta = 1 - \sin^2\theta$	Subtract $\sin^2\theta$ from each side.
$\cos^2\theta = 1 - \left(\frac{3}{5}\right)^2$	Substitute $\frac{3}{5}$ for $\sin\theta$.
$\cos^2\theta = 1 - \frac{9}{25}$	Square $\frac{3}{5}$.
$\cos^2\theta = \frac{16}{25}$	Subtract.
$\cos\theta = \pm\frac{4}{5}$	Take the square root of each side.

Since θ is in the second quadrant, $\cos\theta$ is negative. Thus, $\cos\theta = -\frac{4}{5}$.

b. Find $\csc\theta$ if $\cot\theta = -\frac{1}{4}$ and $270° < \theta < 360°$.

$\cot^2\theta + 1 = \csc^2\theta$	Trigonometric identity
$\left(-\frac{1}{4}\right)^2 + 1 = \csc^2\theta$	Substitute $-\frac{1}{4}$ for $\cot\theta$.
$\frac{1}{16} + 1 = \csc^2\theta$	Square $-\frac{1}{4}$.
$\frac{17}{16} = \csc^2\theta$	Add.
$\pm\frac{\sqrt{17}}{4} = \csc\theta$	Take the square root of each side.

Since θ is in the fourth quadrant, $\csc\theta$ is negative. Thus, $\csc\theta = -\frac{\sqrt{17}}{4}$.

CHECK Your Progress

1A. Find $\sin\theta$ if $\cos\theta = \frac{1}{3}$ and $270° < \theta < 360°$.

1B. Find $\sec\theta$ if $\sin\theta = -\frac{2}{7}$ and $180° < \theta < 270°$.

SIMPLIFY EXPRESSIONS Trigonometric identities can also be used to simplify expressions containing trigonometric functions. Simplifying an expression that contains trigonometric functions means that the expression is written as a numerical value or in terms of a single trigonometric function, if possible.

Study Tip

It is often easiest to write all expressions in terms of sine and/or cosine.

EXAMPLE 2 Simplify an Expression

Simplify $\frac{\csc^2\theta - \cot^2\theta}{\cos\theta}$.

$\frac{\csc^2\theta - \cot^2\theta}{\cos\theta} = \frac{\frac{1}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}}{\cos\theta}$	$\csc^2\theta = \frac{1}{\sin^2\theta}$, $\cot^2\theta = \frac{\cos^2\theta}{\sin^2\theta}$
$= \frac{\frac{1-\cos^2\theta}{\sin^2\theta}}{\cos\theta}$	Add.

$$= \frac{\sin^2\theta}{\frac{\sin^2\theta}{\cos\theta}} \qquad 1 - \cos^2\theta = \sin^2\theta$$

$$= \frac{1}{\cos\theta} \qquad \frac{\sin^2\theta}{\sin^2\theta} = 1$$

$$= \sec\theta \qquad \frac{1}{\cos\theta} = \sec\theta$$

CHECK Your Progress **Simplify each expression.**

2A. $\frac{\tan^2\theta\csc^2\theta - 1}{\sec^2\theta}$

2B. $\frac{\sec\theta}{\sin\theta}(1 - \cos^2\theta)$

EXAMPLE 3 Simplify and Use an Expression

BASEBALL **Refer to the application at the beginning of the lesson. Rewrite the equation in terms of tan θ.**

$h = \left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0$ — Original equation

$= -\frac{16}{v^2}\left(\frac{1}{\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0$ — Factor.

$= -\frac{16}{v^2}\left(\frac{1}{\cos^2\theta}\right)t^2 + (\tan\theta)t + h_0$ — $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$= -\frac{16}{v^2}(\sec^2\theta)t^2 + (\tan\theta)t + h_0$ — Since $\frac{1}{\cos\theta} = \sec\theta$, $\frac{1}{\cos^2\theta} = \sec^2\theta$.

$= -\frac{16}{v^2}(1 + \tan^2\theta)t^2 + (\tan\theta)t + h_0$ — $\sec^2\theta = 1 + \tan^2\theta$

Thus, $\left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0 = -\frac{16}{v^2}(1 + \tan^2\theta)t^2 + (\tan\theta)t + h_0$.

CHECK Your Progress

3. Rewrite the expression $\cot^2\theta - \tan^2\theta$ in terms of $\sin\theta$.

CHECK Your Understanding

Example 1 (p. 838)

Find the value of each expression.

1. $\tan\theta$, if $\sin\theta = \frac{1}{2}$; $90° \leq \theta < 180°$

2. $\csc\theta$, if $\cos\theta = -\frac{3}{5}$; $180° \leq \theta < 270°$

3. $\cos\theta$, if $\sin\theta = \frac{4}{5}$; $0° \leq \theta < 90°$

4. $\sec\theta$, if $\tan\theta = -1$; $270° < \theta < 360°$

Example 2 (pp. 838–839)

Simplify each expression.

5. $\csc\theta\cos\theta\tan\theta$

6. $\sec^2\theta - 1$

7. $\frac{\tan\theta}{\sin\theta}$

8. $\sin\theta(1 + \cot^2\theta)$

Example 3 (p. 839)

9. **PHYSICAL SCIENCE** When a person moves along a circular path, the body leans away from a vertical position. The nonnegative acute angle that the body makes with the vertical is called the *angle of inclination* and is represented by the equation $\tan\theta = \frac{v^2}{gR}$, where R is the radius of the circular path, v is the speed of the person in meters per second, and g is the acceleration due to gravity, 9.8 meters per second squared. Write an equivalent expression using $\sin\theta$ and $\cos\theta$.

Exercises

For Exercises	See Examples
10–17	1
18–26	2
27, 28	3

HOMEWORK HELP

Find the value of each expression.

10. $\tan \theta$, if $\cot \theta = 2$; $0° \leq \theta < 90°$

11. $\sin \theta$, if $\cos \theta = \frac{2}{3}$; $0° \leq \theta < 90°$

12. $\sec \theta$, if $\tan \theta = -2$; $90° < \theta < 180°$

13. $\tan \theta$, if $\sec \theta = -3$; $180° < \theta < 270°$

14. $\csc \theta$, if $\cos \theta = -\frac{3}{5}$; $90° < \theta < 180°$

15. $\cos \theta$, if $\sec \theta = \frac{5}{3}$; $270° < \theta < 360°$

16. $\cos \theta$, if $\sin \theta = \frac{1}{2}$; $0° \leq \theta < 90°$

17. $\csc \theta$, if $\cos \theta = -\frac{2}{3}$; $180° < \theta < 270°$

Simplify each expression.

18. $\cos \theta \csc \theta$

19. $\tan \theta \cot \theta$

20. $\sin \theta \cot \theta$

21. $\cos \theta \tan \theta$

22. $2(\csc^2 \theta - \cot^2 \theta)$

23. $3(\tan^2 \theta - \sec^2 \theta)$

24. $\frac{\cos \theta \csc \theta}{\tan \theta}$

25. $\frac{\sin \theta \csc \theta}{\cot \theta}$

26. $\frac{1 - \cos^2 \theta}{\sin^2 \theta}$

ELECTRONICS **For Exercises 27 and 28, use the following information.**

When an alternating current of frequency f and a peak current I pass through a resistance R, then the power delivered to the resistance at time t seconds is $P = I^2R - I^2R \cos^2 2ft\pi$.

27. Write an expression for the power in terms of $\sin^2 2ft\pi$.

28. Write an expression for the power in terms of $\tan^2 2ft\pi$.

Find the value of each expression.

29. $\tan \theta$, if $\cos \theta = \frac{4}{5}$; $0° \leq \theta < 90°$

30. $\cos \theta$, if $\csc \theta = -\frac{5}{3}$; $270° < \theta < 360°$

31. $\sec \theta$, if $\sin \theta = \frac{3}{4}$; $90° < \theta < 180°$

32. $\sin \theta$, if $\tan \theta = 4$; $180° < \theta < 270°$

Simplify each expression.

33. $\frac{1 - \sin^2 \theta}{\sin^2 \theta}$

34. $\frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta}$

35. $\frac{\tan^2 \theta - \sin^2 \theta}{\tan^2 \theta \sin^2 \theta}$

Real-World Link

The oldest operational carousel in the United States is the Flying Horse Carousel at Martha's Vineyard, Massachusetts.

Source: Martha's Vineyard Preservation Trust

AMUSEMENT PARKS **For Exercises 36–38, use the following information.**

Suppose a child is riding on a merry-go-round and is seated on an outside horse. The diameter of the merry-go-round is 16 meters.

36. Refer to Exercise 9. If the sine of the angle of inclination of the child is $\frac{1}{5}$, what is the angle of inclination made by the child?

37. What is the velocity of the merry-go-round?

38. If the speed of the merry-go-round is 3.6 meters per second, what is the value of the angle of inclination of a rider?

LIGHTING **For Exercises 39 and 40, use the following information.**

The amount of light that a source provides to a surface is called the *illuminance*. The illuminance E in foot candles on a surface is related to the distance R in feet from the light source. The formula $\sec \theta = \frac{I}{ER^2}$, where I is the intensity of the light source measured in candles and θ is the angle between the light beam and a line perpendicular to the surface, can be used in situations in which lighting is important.

39. Solve the formula in terms of E.

40. Is the equation in Exercise 39 equivalent to $R^2 = \frac{I \tan \theta \cos \theta}{E}$? Explain.

EXTRA PRACTICE

See pages 923, 939.

Math Online

Self-Check Quiz at algebra2.com

H.O.T. Problems

41. **REASONING** Describe how you can determine the quadrant in which the terminal side of angle α lies if $\sin \alpha = -\frac{1}{4}$.

42. **OPEN ENDED** Write two expressions that are equivalent to $\tan \theta \sin \theta$.

43. **REASONING** If $\cot(x) = \cot\left(\frac{\pi}{3}\right)$ and $3\pi < x < 4\pi$, find x.

44. **CHALLENGE** If $\tan \beta = \frac{3}{4}$, find $\frac{\sin \beta \sec \beta}{\cot \beta}$.

45. ***Writing in Math*** Use the information on page 837 to explain how trigonometry can be used to model the path of a baseball. Include an explanation of why the equation at the beginning of the lesson is the same as $y = -\frac{16 \sec^2 \theta}{v^2}x^2 + (\tan \theta)x + h_0$.

STANDARDIZED TEST PRACTICE

46. **ACT/SAT** If $\sin x = m$ and $0 < x < 90°$, then $\tan x =$

A $\frac{1}{m^2}$

B $\frac{1 - m^2}{m}$

C $\frac{m}{\sqrt{1 - m^2}}$

D $\frac{m}{1 - m^2}$

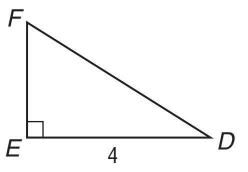
47. **REVIEW** Refer to the figure below. If $\cos D = 0.8$, what is length $\overline{DF}$?

F 5

G 4

H 3.2

J $\frac{4}{5}$



Spiral Review

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function. (Lesson 14-2)

48. $y = \sin \theta - 1$

49. $y = \tan \theta + 12$

Find the amplitude, if it exists, and period of each function. Then graph each function. (Lesson 14-1)

50. $y = \csc 2\theta$

51. $y = \cos 3\theta$

52. $y = \frac{1}{3} \cot 5\theta$

53. Find the sum of a geometric series for which $a_1 = 48$, $a_n = 3$, and $r = \frac{1}{2}$. (Lesson 11-4)

54. Write an equation of a parabola with focus at $(11, -1)$ and directrix $y = 2$. (Lesson 10-2)

55. **TEACHING** Ms. Granger has taught 288 students at this point in her career. If she has 30 students each year from now on, the function $S(t) = 288 + 30t$ gives the number of students $S(t)$ she will have taught after t more years. How many students will she have taught after 7 more years? (Lesson 2-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Name the property illustrated by each statement.** (Lesson1-3)

56. If $4 + 8 = 12$, then $12 = 4 + 8$.

57. If $7 + s = 21$, then $s = 14$.

58. If $4x = 16$, then $12x = 48$.

59. If $q + (8 + 5) = 32$, then $q + 13 = 32$.